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On the Use of Semi-folding in Regular Blocked Two-level Factorial Designs

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In this article, we consider experimental situations where a blocked regular two-level fractional factorial initial design is used. We investigate the use of the semi-fold technique as a follow-up strategy for de-aliasing effects that are confounded in the initial design as well as an alternative method for constructing blocked fractional factorial designs. A construction method is suggested based on the full foldover technique and sufficient conditions are obtained when the semi-fold yields as many estimable effects as the full foldover.

Keywords Foldover design; Minimum aberration; Maximal rank-minimum aberration; Word pattern.

Mathematics Subject Classification Primary 62K15; Secondary 62K05.

1. Introduction

In experimental situations where a two-level fractional factorial (FF) design is initially used to identify influential system variables, it is often necessary to use a follow-up design to increase precision of the treatment effects or gain additional information about the experimental process by de-aliasing effects confounded in the initial design. One type of follow-up strategy mentioned in many textbooks and which has been studied extensively in recent years is the "foldover" technique. In using this technique, a "foldover design" is used reversing the signs of one or more factors in the initial design. By adding the "foldover design" to the initial design, an overall combined design is often obtained which has higher resolution and allows the estimation of more effects than the initial design. The construction of optimal "foldover" designs has been studied by Li and Lin (2003) and Li and Mee (2002) in cases where the initial design was a regular two level FF. More recently, the "foldover" follow-up strategy has been considered in experimental situations where the initial design is a regular blocked two-level FF design, i.e., see Li and Jacroux (2007) and Wu et al. (2010), and the follow-up "foldover" blocked factorial is obtained as described above and has the same blocking scheme as the initial design. However, as pointed out in

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Mee and Peralta (2000), one problem with the foldover technique is that it is very "degrees of freedom inefficient", i.e., if the initial and foldover design each have n runs, then addition of the foldover design provides relatively few additional estimable effects. In fact, Mee and Peralta (2000) found that addition of the foldover design with n runs generally provided fewer than $\frac{n}{2}$ additional degrees of freedom for the estimation of two-factor interactions. To solve this problem, Mee and Peralta (2000) investigated the use of a "semi-fold design" as a follow-up design. A "semi-fold design" is obtained by taking half the runs from the initial design but changing the sign of one or more factors in these runs. Mee and Peralta (2000) found that by appropriately selecting the "semi-fold" follow-up design, the resulting combined design generally yielded as many degrees of freedom for the estimation of two-factor interactions as a corresponding "full foldover" design having n runs. In this paper, we consider the use of the "semi-fold technique" in relation to constructing follow-up designs for blocked regular FF initial designs as well as a method for constructing alternative blocked FF designs. We show that in general, when the number of added factors is not too large, semi-fold follow up designs yield generally as many estimable two-factor interactions as do complete foldover designs. However, when larger numbers of factors are involved, the semi-fold process typically allows for the estimability of fewer two-factor interactions than does a full foldover design.

2. Notation and Definition

In this section, we give the basic definitions and notation that are used throughout the sequel.

We shall henceforth represent an arbitrary two-level FF design d by an $n \times m$ matrix $X_d = (x_{d_1}, \cdots, x_{d_m})$ whose columns x_{d_i} have entries $+1$ or -1. Each row of X_d corresponds to a run in d and each column to an experimental factor. An orthogonal two-level main effects design satisfies $X_d' X_d = n\ I_m$ where X_d' denotes the transpose of X_d and I_m is the $m \times m$ identity matrix.

In this article, we will be considering what are typically referred to as regular 2^{m-k} FF designs. A 2^{m-k} regular FF design has m factors and 2^{m-k} runs. Of the m factors, there are $m - k$ factors, which we shall assume are labeled $1, \cdots, m - k$, which are called basic factors and are such that the design contains a complete factorial in these factors. The other k factors, labeled $m - k + 1, \cdots, m$, are called added factors and are obtained by associating with each added factor an interaction involving basic factors, i.e., for added factors $l = m - k + 1, \cdots, m$, $l = l_1 \cdots l_t$ where $l_1, \cdots, l_t$ denote basic factors. For $l = m - k + 1, \cdots, m$, the strings of factor labels $l_1 \cdots l_t l$ are called treatment defining effects words. The group formed by taking all possible products among the treatment defining effects words (according to the rule that if a factor label appears an even number of times in the product it is eliminated whereas if it appears an odd number of times it is kept) is called the treatment defining relations group which we denote by $G_t(d)$. Including I, the identity element, a 2^{m-k} design d has 2^k words in $G_t(d)$ and the number of factor labels in a word is called the length of the word.

For given values of m and k, there are typically a large number of 2^{m-k} regular FF designs that can be constructed using different defining relations. To aid in the construction of "good" designs, the criteria of resolution and minimum aberration (MA) were introduced. The resolution of a given design d is given by the length of the shortest word in $G_t(d)$. However, there are often a number of 2^{m-k} desgins having the same resolution. To select among the designs having the same resolution a best design, Fries and Hunter (1986) proposed a refinement of the resolution criterion which they called MA. For a 2^{m-k} design d, let $W_{t_i}(d)$ be the number of words of length i in $G_t(d)$. Then $W_t(d) = (W_{t_3}(d), \cdots, W_{t_m}(d))$ is called the treatment word length vector of the design. Now, for two designs $d_1$and d_2, let r be the smallest integer such that $W_{t_r}(d_1) \neq W_{t_r}(d_2)$. Then $d_1$is said to have less aberration than d_2 if $W_{t_r}(d_1) < W_{t_r}(d_2)$. If no design has less aberration than d_1, then $d_1$is called an MA design. Chen et al. (1993) provided a catalog containing many MA designs for various values of m and k.

In many experimental situations where a 2^{m-k} design is appropriate, blocking is an effective method for improving the efficiency of an experiment by eliminating sources of heterogeneity. Blocking can be accomplished in a regular FF design through the use of blocking factors which are obtained in much the same manner as the added treatment factors described previously. In particular, a blocking factor b_i for $i = 1, \cdots, p$ is obtained by associating b_i with an interaction $i_1 \cdots i_t$ among the basic factors i_j which has not already been associated with an added treatment factor. The set of all products that can be formed between words $i_1 \cdots i_t b_i$ for $i = 1, \cdots, p$ by the same product rule as described for $G_t(d)$ is called the block defining relations group of d and is denoted by $G_b(d)$. Finally, the set of all products that can be formed between words in $G_t(d)$ and $G_b(d)$ is denoted by $G_{t\times b}(d)$. We will call $G(d) = G_t(d) \cup G_b(d) \cup G_{t\times b}(d)$ the defining relations group for a blocked 2^{m-k} design d and we shall denote a regular 2^{m-k} design that is blocked in 2^p blocks as a $2^{(m+p)-(p+k)}$ design. For a $2^{(m+p)-(p+k)}$ design d, we use $W_{b_i}(d)$ to denote the number of words in $G_b(d) \cup G_{t\times b}(d)$ containing i treatment letters and call $W_b(d) = (W_{b_2}(d), \cdots, W_{b_m}(d))$ the block word length vector of d.

Throughout this article, we will consider the situation where a $2^{(m+p)-(p+k)}$ design is to be used and where the experimenter is interested in obtaining as much information on treatment effects and two-factor interactions as possible. We will only be considering situations in which no main effect is aliased with another main effect or block effect. When analyzing the data from a given $2^{(m+p)-(p+k)}$ regular design d, we will assume that three-factor and higher-order interactions are negligible. Within this context, for given values of m, k, and p, and a given design d having $X_d = (x_{d_1}, \cdots, x_{d_m})$, the model for analysis is

$$Y = X_{d_1}\beta_1 + X_{d_2}\beta_2 + X_{d_3}\beta_3 + \varepsilon, \tag{2.1}$$

where Y is a $2^{m-k} \times 1$ vector of observations, $X_{d_1} = (1_{2^{m-k}}, X_d)$, 1_p is a $p \times 1$ vector of 1's, $\beta_1' = (\beta_0, \beta_1, \cdots, \beta_m)$ where β_0 represents an overall mean, $\beta_1, \cdots, \beta_m$ are the main effect parameters, β_2 is the vector of $\binom{m}{2}$ two-factor interaction parameters, X_{d_2} is the corresponding two-factor interaction matrix obtained by taking Hadamard products of all pairs of columns in X_d, β_3 is the vector of block parameters having block design matrix X_{d_3}, and ε is a vector of uncorrelated random error terms assumed to have mean 0 and constant variance σ^2.

3. Foldovers for 2^{m-k} and $2^{(m+p)-(p+k)}$ designs

In many experimental settings, once a screening experiment has been performed and possible significant experimental effects identified, a standard follow-up strategy discussed in many textbooks involves adding a second fraction to help dealias effects associated with significant contrasts determined from the initial experiment. One type of follow up design often suggested for usage in such situations is a foldover design which is obtained by reversing the signs of one or more columns in X_d of the initial design d. In a 2^{m-k} design, there are 2^m possible ways to generate a foldover design.

With regard to foldover plans for regular 2^{m-k} designs, we will adopt much of the notation used in Li and Lin (2003). In particular, we will let δ denote a foldover plan where δ is a listing of the columns of X_d whose signs are to be reversed in the foldover design, then each foldover design is generated by a foldover plan. The design obtained by joining the foldover design to the initial design is called the combined design. We will denote the initial design by d, the foldover design by d', and the combined design obtained by joining d and d' by $\mathrm{D} = \binom{d}{d'}$.

The foldover process as applied to $2^{(m+p)-(p+k)}$ designs is similar to that described above for $2^{(m-k)}$ designs. When applying the foldover process to a given $2^{(m+p)-(p+k)}$ design d, we will use the same notation as described previously for a foldover of a regular $2^{(m-k)}$ design. We will let δ denote the list of columns from X_d whose signs are reversed to obtain the foldover design denoted by d'. As in Li and Jacroux (2007) and Wu et al. (2010), we will assume the same blocking factors are used to obtain blocks in d' as were used in d and denote the combined design by D. Under these assumptions, we note that $G_b(d)$ and $G_b(d')$ are both the same whereas $\mathrm{G_b(D)}$ consists of all those words in $\mathrm{G_b(d)}$ along with a treatment factor generator word from $\mathrm{G_t(d)}$ (and all its products with words in $\mathrm{G_b(d)}$) that is eliminated through the foldover process, i.e., one the the treatment factor generator words in $\mathrm{G_t(d)}$ becomes a block generator word in $\mathrm{G_b(D)}$. We shall also use notation such as $W_{t_i}(d)$, $W_{t_i}(d')$, and $W_{t_i}(D)$ to denote the number of words of length i in $G_t(d)$, $G_t(d')$, and $G_t(D)$, respectively, for $i = 3, \cdots$,m and $W_{b_i}(d)$, $W_{b_i}(d')$, and $W_{b_i}(D)$ to denote the number of words having i treatment factor labels in them in $G_b(d) \cup G_{b\times t}(d)$, $G_b(d') \cup G_{b\times t}(d')$, and $G_b(D) \cup G_{b\times t}(D)$, respectively, for $i = 2, \cdots$,m and finally let $W_t(d)$, $W_t(d')$, $W_t(D)$, $W_b(d)$,$W_b(d')$, and W_b(D) denote the various word length vectors corresponding to d, d' and D.

For constructing optimal foldover plans for a given $2^{(m+p)-(p+k)}$ design d, we consider the primary optimality criterion used in Li and Lin (2003) and Li and Jacroux (2007). This consists of applying the MA criterion to the defining relations group of D. So for a given $2^{(m+p)-(p+k)}$ design d, let δ be a foldover plan, let D(δ) be the combined design, and let W_{t_i}(D(δ)) denote the number of words of length i in G_t(D(δ)). The MA criterion is then defined as follows.

Let δ_1 and δ_2 be two foldover plans. We say δ_1 has less aberration than δ_2 if $W_{t_i}(\mathrm{D}(\delta_1)) = W_{t_i}(\mathrm{D}(\delta_2))$ for i = 3,$\cdots$,r-1, and $W_{t_r}(\mathrm{D}(\delta_1)) < W_{t_r}(\mathrm{D}(\delta_2))$. If no foldover plan has less aberration than δ_1, we call δ_1 an MA optimal foldover plan for d, which we denote by δ^*_{MA}.

With regard to actually searching for MA-optimal foldovers, Li and Lin (2003) proved that for regular FF designs, the actual search could be limited to foldovers only involving added factors which they called “core” foldovers as all other foldovers are equivalent to some “core” foldover. It is easily seen using similar arguments that the same holds for blocked regular FF designs. Using this fact and computer search methods, Li and Jacroux (2007) found the MA optimal foldover plans for various values of m, p, and k and run sizes of 16, 32, and 64. Within the context of these MA foldovers, we consider the construction of semi-foldover designs.

4. Semi-Foldovers

As pointed out in the Introduction, adding a foldover design to an initial regular 2^{m-k} design is often inefficient in terms of increasing the number of estimable two-factor interactions in the combined design. In fact, Mee and Peralta (2000) found that for most resolution 4 2^{m-k}designs, it is possible to add half the observations from a foldover design to an initial design and still realize the same increase in number of estimable two-factor interactions as with adding the full foldover design. The technique of adding half the observations from a foldover design to the initial design is called a semi-foldover and the observations added is termed a semi-foldover design.

In general, the authors have found that the semi-foldover technique does not work quite as well when applied to 16-run blocked regular FF design as it does for regular FF designs in terms of generating estimable two-factor interactions but that the technique is quite effective when applied to blocked FF designs having 32 and 64 runs. In this paper, we explore under what conditions the semi-foldover process is successful in increasing the number of estimable two-factor interactions to the same level as in the full foldover process. We begin with an example.

Example 4.1 Consider the case where an initial $2^{(6+2)-(2+2)}$ design d is to be run[1]. The design that would be recommended in many textbooks has $G_t(d) = \{1235, 1246, 3456\}$ and $G_b(d) = \{134, 234, 12\}$. An MA optimal foldover design d' given in Li and Jacroux (2007), obtained by folding over factors 5 and 6 in d and maintaining the same blocking scheme would have $G_t(d') = \{-1235, -1246, 3456\}$ and $G_b(d') = \{134, 234, 12\}$. The combined design $D = (\begin{smallmatrix} d \\ d' \end{smallmatrix})$ would have $G_t(D) = \{3456\}$ and $G_b(D) = \{134, 234, 12, 1235, 245, 145, 35\}$. We note that the treatment factor generator word 1235 in d becomes a block generator word in D. In the combined design D, two-factor interactions 12, 35, and 46 are all aliased with blocks in d and d', and hence are nonestimable in D. Also, we have alias sets $34 = 56$ and $36 = 45$ that are the same in both d and d', and hence are the same in D. All other interactions become estimable after adding d' to d, thus all main effects and 10 out of 15 two-factor interactions are estimable in D. There is an alternative way of viewing the above foldover process. We consider this alternative view because (1) it provides a basis for the semi-foldover technique suggested later by the authors and (2) it allows for the development of a systematic method for finding the partial confounding scheme (through a set of weighted defining effect words) associated with the semi-foldover combined designs generated using the authors' suggested construction method. In particular, consider the following.

1. From the original design in d, identify one of the block generators, say $b_1 = 134$, as an added factor generator (thus leaving the blocking generator to be 234) in the $2^{(6+1)-(1+3)}$ design *d(1)* having $G_t(d(1)) = \{\begin{smallmatrix} 1235,\ 1246,\ 3456 \\ 134,\ 245,\ 236,\ 156 \end{smallmatrix}\}$ and $G_b(d(1)) = \{234\}$.
2. To obtain a design which is essentially equivalent to d above, foldover factors 4 and 6 in $d(1)$ to obtain d(2) having $G_t(d(2)) = \{\begin{smallmatrix} 1235,\ 1246,\ 3456 \\ -134,\ -245,\ -236,\ -156 \end{smallmatrix}\}$ and $G_b(d(2)) = \{234\}$. The reason for folding overs factors 4 and 6 is to generate *d(2)* such that its first row is exactly the same as *d(1)* and its second row has signs opposite to those in *d(1)* such that the combined design $D(1) = (\begin{smallmatrix} d(1) \\ d(2) \end{smallmatrix})$ has $G_t(D(1)) = \{1235, 1246, 3456\}$ and $G_b(D(1)) = \{234, 134, 12\}$. We note that $D(1)$ has exactly the same alias structure as d and exactly the same estimable main effects and two factor interactions.

[1]Basic factors: 1, 2, 3, 4; added factors: $5 = 123$, $6 = 124$; and blocking factors: $b_1 = 134$ and $b_2 = 234$.

3. To obtain a design $D(2)$ which is essentially equivalent to D in the previous foldover process, foldover factors 5 and 6 in $D(1)$ to obtain $D(1)'$ and let $D(2) = \begin{pmatrix} D(1) \\ D(1)' \end{pmatrix}$. Unlike the foldover plan in step 2, the foldover plan in this step is a MA optimal foldover plan given in Li and Jacroux (2007). We note D(2) and D have exactly the same alias structure and the same estimable main effects and two factor interactions. We also note that applying the foldover process to $D(1) = \begin{pmatrix} d(1) \\ d(2) \end{pmatrix}$ to get $D(2)$ is equivalent to folding over factors 5 and 6 in d(1) and $d(2)$ and getting $d(1)'$ and $d(2)'$ having $G_t(d(1)') = \{134, -1235, -1246, -245, -236, 3456, 156\}$, $G_b(d(1)') = \{234\}$ and $d(2)'$ having $G_t(d(2)') = \{-134, -1235, -1246, 245, 236, 3456, -156\}$, $G_b(d(2)') = \{234\}$, and
$$D(2) = \begin{pmatrix} d(1) \\ d(2) \\ d(1)' \\ d(2)' \end{pmatrix}.$$

At this point, we observe that if in $D(2)$ we only keep the runs corresponding to $d(1)$, $d(2)$, and $d(1)'$, then the resulting design $\tilde{D} = \begin{pmatrix} d(1) \\ d(2) \\ d(1)' \end{pmatrix}$ has the same estimable main effects and two-factor interactions as does $D(2)$, but the estimates are not orthogonal to one another as they are in $D(2)$. Thus, in this case, adding the semi-foldover design $d(1)'$ to $D(1)$ to get $\tilde{D}$ is as effective as adding all the observations in $D(1)'$ to obtain D (2).

To obtain the full and partial aliasing scheme for effects in $\tilde{D}$ we consider a weighted treatment defining relations group for $\tilde{D}$. To obtain this weighted treatment group, we observe that

$$\begin{aligned} G_t(d(1)) &= \{134, 1235, 1246, 245, 236, 3456, 156\} \\ G_t(d(2)) &= \{-134,\ 1235,\ 1246,\ -245,\ -236,\ 3456, -156\} \\ G_t(d(1)') &= \{134, -1235, -1246, -245, -236,\ 3456,\ 156\} \end{aligned}$$

and assign a weight of $\mp\frac{1}{3}$ to each word in $G_t(d(1))$, $G_t(d(2))$, and $G_t(d(1)'$ depending on whether the word has a ∓ 1 sign in front of it. We then add the corresponding words together to obtain the weighted group $G_t(\tilde{D}) = \{(\frac{1}{3})134, (\frac{1}{3})1235, (\frac{1}{3})1246, (-\frac{1}{3})1245, (-\frac{1}{3})236, 3456, (\frac{1}{3})156\}$. We observe that other than the weights, the effects appearing in $G_t(\tilde{D})$ are exactly the same as the effects appearing in $G_t(d(1))$. The reason that the weight of $\mp\frac{1}{3}$ is assigned to each word in G_t(d(1)), G_t(d(2)), and G_t(d(1)') is so that when d(1), d(2), and d(1)' are added together to get $\tilde{D}$, the weights associated with the corresponding words in $G_t(\tilde{D})$ reflect the amount of confounding between the main effects and interactions associated with that word. For example, the word 1235 in $G_t(\tilde{D})$ has a weight of 1/3 indicating that the pairs of effects $\{12, 35\}$, $\{13, 25\}$, and $\{15, 23\}$ are only 1/3 partially confounded with each other. On the other hand, the word 3456 in $G_t(\tilde{D})$ has a weight of 1 indicating that the pairs of effects $\{34, 56\}$, $\{35, 46\}$, and $\{36, 45\}$ are fully confounded with one another in $\tilde{D}$. Using this weighting method, to obtain the effects confounded with any main effect or two-factor interaction in $\tilde{D}$, we simply multiply that effect by all of the words and their weights in $G_t(\tilde{D})$. For example, the effects confounded with main effect 1 are

$$1 = \left(\frac{1}{3}\right) 34 = \left(\frac{1}{3}\right) 235 = \left(\frac{1}{3}\right) 246 = \left(-\frac{1}{3}\right) 245 = \left(-\frac{1}{3}\right)$$
$$1236 = 13456 = \left(\frac{1}{3}\right) 56.$$

Any interaction in the alias set of 1 with a weight of $\mp\frac{1}{3}$ indicates that 1 is partially confounded with that interaction whereas any interaction with a weight of 1 indicates that main effect 1 is totally confounded with that interaction. From this, we see that 1 is partially confounded with two-factor interactions 34 and 56. Again we note that since the effects occurring in the weighted group $G_t(\tilde{D})$ and $G_t(d(1))$ are the same, the alias set corresponding to 1 is the same in both $\tilde{D}$ and $d(1)$ with the exception of the assigned weights in $\tilde{D}$. The meaning of the above set of weighted aliases in terms of the information matrix for main effects and two-factor interactions in model (2.1) is that the column of $X_{\widetilde{D_1}}$ corresponding to main effect 1 has an inner product of 8 with the columns of $X_{\widetilde{D_2}}$ corresponding to interaction 34 and 56 and 0 inner-product with all other columns. Similarly, the weighted alias set for 36 is

$$36 = \left(\frac{1}{3}\right) 146 = \left(\frac{1}{3}\right) 1256 = \left(\frac{1}{3}\right) 1234 = \left(-\frac{1}{3}\right) 123456 = \left(-\frac{1}{3}\right)$$
$$2 = 45 = \left(\frac{1}{3}\right) 135$$

from which we see that 36 is partially confounded with main effect 2 and is fully confounded with two-factor interaction 45. The meaning of this weighted alias set in terms of the information matrix for main effects and two-factor interactions is as above with main effect 1. Overall, the information matrix for main effects and two-factor interactions under model (2.1) and $\tilde{D}$ will consist of a series of 3×3 matrices on the main diagonal (one for each weighted alias set) and zeros elsewhere. We do not include columns in $X_{\widetilde{D_2}}$ of model (2.1) for two-factor interactions 12, 35 and 46 because they are completely confounded with blocks in $d(1)$, $d(2)$, and $d(1)'$ and hence are nonestimable. The total number of estimable main effects and two-factor interactions in $\tilde{D}$ is simply the sum of the ranks of the square matrices appearing on the main diagonal of the information matrix of $\tilde{D}$ for main effects and two-factor interactions which is easily seen to be 18.

Comment. As noted previously, there is often more than one foldover plan which is MA optimal for a given $2^{(m+p)-(p+k)}$ design d. In the above example, folding over factor 5, factor 6, or factors 5 and 6 together all lead to MA optimal combined designs. However, in terms of the semi-foldover plan given above, all these optimal foldover plans yield the same number of estimable main effects an two-factor interactions when considered in the context of the semi-foldover design.

We note that in general, given an initial $2^{(m+p)-(p+k)}$ design d, there is more than one choice of block generator in $G_b(d)$ to identify as an added factor in the reduced $2^{(m+p-1)-(p-1+k+1)}$ design as $d(1)$ in the previous example. In general, these different choices lead to different semi-foldover designs having different estimability properties. We illustrate this by continuing Example 4.1.

Example 4.1. Continued.

1. Suppose in the original design d, we select 12 from $G_b(d)$ as an added factor generator to obtain the $2^{[6+(2-1)]-[(2-1)+(2+1)]}$ design $\overline{d(1)}$ having $G_t(\overline{d(1)}) = \{12, 1235, 1246, 35, 46, 3456, 124546\}$, and $G_b(\overline{d(1)}) = \{234\}$.
2. From $\overline{d(1)}$, for the same purpose as we showed in step 2 in the earlier example, we foldover factors 2, 5, and 6 in $\overline{d(1)}$ to obtain $\overline{d(2)}$ having $G_t(\overline{d(2)}) = \{-12, 1235, 1246, -35, -46, 3456, -123456\}$, and $G_b(\overline{d(2)}) = \{234\}$. Then the combined design $\overline{D(1)} = (\dfrac{\overline{d(1)}}{\overline{d(2)}})$ has $G_t(\overline{D(1)}) = \{1235, 1246, 3456\}$ and $G_b(\overline{D(1)}) = \{12, 134, 234\}$. We note that $\overline{D(1)}$ has exactly the same alias structure as d and exactly the same estimable main effects and two-factor interactions. For example, 12, 35, and 46 are not estimable in $\overline{D(1)}$ because they are completely confounded with the mean in $\overline{d(1)}$ and $\overline{d(2)}$.
3. To obtain the semi-foldover design $\tilde{\tilde{D}}$ corresponding to $\overline{d(1)}$, we let $\tilde{\tilde{D}} = \begin{pmatrix} \overline{d(1)} \\ \overline{d(2)} \\ \overline{d(1)}' \end{pmatrix}$ where $\overline{d(1)}'$ is obtained from $\overline{d(1)}$ by folding over factors 5 and 6 in $\overline{d(1)}$.

We note that $\tilde{\tilde{D}}$ above has fewer estimable main effects and two-factor interactions than does $\tilde{D}$. To see this, we obtain the weighted treatment defining relations group for $\tilde{\tilde{D}}$ as above:

$$
\begin{aligned}
G_t\left(\overline{d(1)}\right) &= \{12, 1235, 1246, 35, 46, 3456, 123456\} \\
G_t\left(\overline{d(2)}\right) &= \{-12, 1235, 1246, -35, -46, 3456, -123456\} \\
G_t\left(\overline{d(1)}'\right) &= \{12, -1235, -1246, -35, -46, 3456, 123456\}.
\end{aligned}
$$

Now, assigning a weight of $\pm\frac{1}{3}$ to each word in $G_t(\overline{d(1)})$, $G_t(\overline{d(2)})$, and $G_t(\overline{d(1)}')$ and adding the corresponding weighted words, we obtain the weighted group

$$
G_t\left(\tilde{\tilde{D}}\right) = \left\{\left(\frac{1}{3}\right)12, \left(\frac{1}{3}\right)1235, \left(\frac{1}{3}\right)1246, \left(\frac{1}{3}\right)35, \left(-\frac{1}{3}\right)46, 3456, \left(\frac{1}{3}\right)123456\right\}.
$$

The weighted alias classes of $G_t(\tilde{\tilde{D}})$ (excluding all third and higher order interactions) are

$$
1 = \left(\tfrac{1}{3}\right)2, 3 = \left(-\tfrac{1}{3}\right)5, 4 = \left(-\tfrac{1}{3}\right)6, 13 = \left(\tfrac{1}{3}\right)23 = \left(\tfrac{1}{3}\right)25 = \left(-\tfrac{1}{3}\right)15, 14 = \left(\tfrac{1}{3}\right) \\
24 = \left(\tfrac{1}{3}\right)26 = \left(-\tfrac{1}{3}\right)16, 34 = \left(-\tfrac{1}{3}\right)45 = \left(-\tfrac{1}{3}\right)36 = 56.
$$

We note that 12, 35, and 46 are completely confounded with the mean in $\overline{d(1)}$, $\overline{d(2)}$, and $\overline{d(1)}'$) hence are not estimable. From the above weighted alias sets, the number of estimable main effects and two-factor interactions is again the sum of the ranks of the squared matrices occurring on the main diagonal of the information matrix for main effects and two-factor interactions under model (2.1) and $\tilde{\tilde{D}}$ which is easily seen to be 14. Thus we see that $\tilde{D}$ yields more estimable main effects and two-factor interactions than does $\tilde{\tilde{D}}$.

Comment. We again observe that all of the MA optimal foldover plans when applied to $\overline{d(1)}$ in the above example result in the same number of estimable effects in the corresponding semi-foldover design.

Based on the previous example, we suggest the following procedure for constructing a follow-up semi-foldover design.

1) Select an initial $2^{(m+p)-(p+k)}$ design d using some optimality criterion.
2) From G_b(d), select a generator $l_1 \cdots l_t b_j = g_1$ and from g_1 identify a basic factor from d, say $m-k$, as an added factor and then use g_1 (after eliminating b_j from the word) along with the other generators of d in G_t(d) to form a new $2^{[m+(p-1)]-[(p-1)+(k+1)]}$ design $d(1)$. We observe that $G_b(d(1))$ will have all words in it that can be obtained by taking products of generators in G_b(d),other than g_1, i.e., $G_b(d(1))$ is what is left in $G_b(d)$ after eliminating all words in $G_b(d)$ containing g_1 in their generator product.
3) Construct the combined design $\tilde{D} = \begin{pmatrix} d(1) \\ d(2) \\ d(1)' \end{pmatrix}$, where: (4.2)
   a. $d(2)$ is obtained from $d(1)$ by folding over $m-k$ and all other added factor generators in $d(1)$ that contain $m-$k as part of their generating interaction; and
   b. $d(1)'$ is obtained from $d(1)$ by folding over those added factors that correspond to one of the MA optimal foldover plans as given in Li and Jacroux (2007) corresponding to d.

Comment. We note that in the construction process (4.2) just described, all of $d(1)$, $d(2)$, and $d(1)'$ have the same blocking scheme.

Example 4.2

1. Consider a $2^{(4+2)-(2+3)}$ design[2] with G_t(d) $= \{1235, 1246, 1347, 3456, 2457, 2367, 1567\}$ and G_b(d) $= \{12, 13, 23\}$.
2. Select generator $g_1 = 12$ from G_b(d) and form the $2^{[4+(2-1)]-[(2-1)+(3+1)]}$ design *d(1)* having $G_t(d(1)) = \left\{ \begin{matrix} 1235, 1246, 1347, 3456, 2457, 2367, 1567 \\ 12, 35, 46, 2347, 123456, 1457, 1367, 2567 \end{matrix} \right\}$. Observe that the first row in $G_t(d(1))$ is obtained from $G_t(d)$ and the second row from multiplying G_t(d) by g_1.
3. Construct the combined design $\tilde{D} = \begin{pmatrix} d(1) \\ d(2) \\ d(1)' \end{pmatrix}$.
   a) Foldover factors 1, 3, and 6 in *d(1)* and obtain *d(2)* with$G_t(d(2)) = \left\{ \begin{matrix} 1235, 1246, 1347, 3456, 2457, 2367, 1567 \\ -12, -35, -46, -2347, -123456, -1457, -1367, -2567 \end{matrix} \right\}$ such that $G_t(d(2))$ differs from $G_t(d(1))$ only by the signs in its second row. Factors 1, 3, and 6 were folded over for the sign-changing purpose as in Example 4.1.
   b) Fold over factors 5, 6, and 7 in *d(1)* and obtain $d(1)'$ with$G_t(d(1)') = \left\{ \begin{matrix} -1235, -1246, -1347, 3456, 2457, 2367, -1567 \\ 12, -35, -46, -2347, 123456, 1457, 1367, -2567 \end{matrix} \right\}$. Folding over factors 5, 6, and 7 corresponds to an MA optimal foldover plan as given in Li and Jacroux (2007).
   c) Combine *d(1)*, *d(2)*, and $d(1)'$ and form $\tilde{D}$. Add a weight of $\mp\frac{1}{3}$ to each word of $G_t(d(1))$, $G_t(d(2))$, and $G_t(d(1)'$. Add the corresponding words together and obtain the weighted group $G_t(\tilde{D}) = \{(\frac{1}{3})1235, (\frac{1}{3})1246, (\frac{1}{3})1347, 3456, 2457, 2367, (\frac{1}{3})1567, (\frac{1}{3})12, (-\frac{1}{3})35, (-\frac{1}{3})46, (-\frac{1}{3})2347, (\frac{1}{3})123456, (\frac{1}{3})1457, (\frac{1}{3})1367\}$.

[2]Basic factors: 1, 2, 3, 4; added factors: $5 = 123$, $6 = 124$, and $7 = 134$; and blocking factors: $b_1 = 12$ and $b_2 = 13$.

d) Using $G_t(\tilde{D})$, the sets of effects that are fully and partially confounded with one another are easy to determine and that the number of estimable main effects and two factor interactions in $\tilde{D}$ is 13.

With regard to the construction process outlined in (4.2) above, there are essentially two questions to consider once the initial $2^{(m+p)-(p+k)}$ design d is selected.

1. Which generator from $G_b(d)$ should be used to form the $2^{[m+(p-1)]-[(p-1)+(k+1)]}$ design $d(1)$ (along with which basic factor in this generator to identify as an added factor in $d(1)$.)
2. Which MA optimal foldover plan given in Li and Jacroux (2007) for d to apply to d(1) to optimize the number of estimable main effects and two-factor interactions in $\tilde{D} = \begin{pmatrix} d(1) \\ d(2) \\ d(1)' \end{pmatrix}$.

With regard to the above questions, we make the following observations.

1. The choice of a generator from d to form $d(1)$ can make a difference as illustrated in Example 4.1. Thus it may be necessary to try all possible generators from $G_b(d)$ to find the one which optimizes the number of estimable main effects and two-factor interactions in $\tilde{D}$ of (4.2).
2. Once the generator from $G_b(d)$ has been selected, any of the basic factors from d in the generator can be selected as an added factor in $d(1)$. Without loss of generality, let m-k denote the basic factor in d selected as an added factor in $d(1)$. To obtain d from $d(1)$, simply form $d(2)$ by folding over added factor m-k in $d(1)$ along with all other added factors in $d(1)$ that contain factor m-k in their generator interaction. The design $\begin{pmatrix} d(1) \\ d(2) \end{pmatrix}$ then has the same treatment defining relations group as d as well as essentially the same blocking scheme.
3. To obtain $d(1)'$ from $d(1)$ in construction process (4.2), the authors have found by exhaustive search that applying any of the MA-optimal foldover plans to $d(1)$ to get $d(1)'$ yields the same number of estimable effects for $\tilde{D}$ in (4.2). A theoretical proof of this finding has not been found.
4. All two-factor interactions that are confounded with block effects or the overall mean in $d(1)$ remain confounded in $\tilde{D}$ and are not estimable in $\tilde{D}$. We also note that, as in Example 4.1, the effects in $G_t(d(1))$ and the weighted class $G_t(\tilde{D})$ are exactly the same.

Using the above construction process (4.2), we now give a Proposition which provides sufficient conditions for the semi-foldover design $\tilde{D}$ to yield as many estimable main effects and two-factor interactions as the full foldover design D. However, before giving the Proposition, we introduce some additional notation.

Consider the $2^{[m+(p-1)]-[(p-1)+(k+1)]}$ design $d(1)$, the $2^{[m+(p-1)]-[(p-1)+(k+1)]}$ design D and $\tilde{D}$ in construction process (4.2). We have the following easily established facts concerning the alias sets in each of the designs.

1. The alias sets in $d(1)$ are exactly the same as the "weighted" alias sets in $\tilde{D}$. The only difference is that all interactions in an alias set of $d(1)$ are totally confounded with each $\{\{$other while an interaction in an alias set of $\tilde{D}$ may be totally or partially confound with other interactions in the alias set.
2. If two interactions are totally confounded in $\tilde{D}$, then they are totally confounded in D and members of the same alias set in D.

3. Each alias set in *d(1)* (hence in $\tilde{D}$) of size 2^{k+1} is the union of four mutually exclusive alias sets of size 2^{k-1} from the full foldover design D.

Example 4.2. Continued. To illustrate fact 3 just given above, observe that the alias set of interaction 14 in d(1) consists of the effects {14,2345,26,37,1356,1257,123467,4567,24,1345,16,1237,2356,57,3467,124567}. The full foldover design D obtained after folding over factors 4, 5 and 6 in d has $G_t(D) = \{3456,2457,2367\}$. Hence, we see that the alias set of interaction 14 in d(1) is the union of the four alias sets {14,1356,1257,123467}, {26,2345,4567,37}, {24,2356,57,3467}, {16,1345,124567,1237} from D.

Using these facts, we establish the following proposition.

Proposition 4.1 In construction process (4.2), consider the designs $d(1)$, $\tilde{D}$ and D as above. If each alias set in $d(1)$ containing a main effect or two-factor interaction is the union of four alias sets from D, at most three of which contain a main effect or interaction, then $\tilde{D}$ allows for estimation of as many main effects and interactions as does D.

Proof. In Model (2.1) under D, the number of estimable main effects and two factor interactions in D is the same as the number of the 2^{m-k+1} alias sets in D that contain a main effect or two-factor interaction. Now consider design $\tilde{D}$. Each "weighted" alias set in $\tilde{D}$ has the same members as the corresponding alias set in $d(1)$. Consider model (2.1) under $\tilde{D}$. Let α be a main effect or 2-factor interaction in $\tilde{D}$ and let $A_\alpha(\tilde{D})$ be its corresponding "weighted" alias set and note that, as observed above,

$$A_\alpha\left(\tilde{D}\right) = \cup_{i=1}^{4}\ A_{B_i}\left(D\right),\ \text{where}$$

B_i is an interaction in D and $A_{B_i}(D)$ is its corresponding alias set in D. Then only one element from each $A_{B_i}(D)$ is estimable in D and that single element is partially confounded with α as well as partially confounded with any other interaction from a different alias set. Now, under the assumption in the Proposition, at most three of the alias sets from $A_{B_i}(D)$ contain a main effect or two factor interaction. Without loss of generality, assume there are three such alias sets and they are $A_{B_i}(D)$, i = 1, 2, 3 and from each such alias set, select $\gamma_1 = \alpha$ and γ_2, γ_3 which are either main effects or two factor interactions. Let X_{γ_1}, X_{γ_2}, X_{γ_3} be the columns in Model (2.1) under $\tilde{D}$ corresponding to X_{γ_1}, X_{γ_2}, X_{γ_3}. Then we note that

$$X'_{\gamma_i}X_{\gamma_i} = 2^{(m+p)-(p+k)} + 2^{(m+p)-(p+k+1)}\ \text{and}$$
$$\left|X'_{\gamma_i}X_{\gamma_j}\right| = 2^{(m+p)-(p+k+1)}\ \text{fori, j} = 1,\ 2,\ 3,\ \text{i} \neq \text{j}.$$

Using these facts, it follows that the 3×3 matrix on the main diagonal of the reduced normal equations for main effects and two-factor interaction corresponding to γ_1, γ_2, and γ_3 under $\tilde{D}$ has off diagonal elements such that the sum of the absolute values of the off-diagonal elements in each row is smaller than the corresponding diagonal element. Such matrices are well known to be nonsingular, e.g., see Graybill (2001), hence each of γ_1, γ_2, and γ_3 are estimable in $\tilde{D}$ as well as D. Since a similar argument can be made for each such weighted alias set $A_\alpha\left(\tilde{D}\right) = \cup_{i=1}^{4}\ A_{B_i}(D)$ and each number 1, 2, or 3 of the $A_{B_i}(D)$ containing a main effect or two-factor interaction, it follows that the number of main effects and two-factor interactions estimable in both $\tilde{D}$ and D is the same.

Using the sufficient conditions given in the previous proposition, it is relatively simple to identify which $2^{(m+p)-(p+k)}$ designs are such that D and $\tilde{D}$ have the same number of estimable main effects and two-factor interactions.

5. The Tables

In the Appendix, we give Tables 1, 2, and 3 corresponding to regular blocked FF designs having 16, 32, and 64 runs, respectively. The first column of each table gives the same labeling to the initial $2^{(m+p)-(p+k)}$ design being considered as given in Sun et al. (1997). The second and fourth columns give the word generators for $G_t(d)$ and $G_b(d)$ whereas the third and fifth columns give the word length pattern vectors of $G_t(d)$ and $G_b(d) \cup {}^{G}_{t\times b}(d)$. In the sixth column, the MA-optimal foldovers δ_{MA} are given and the word length vector for $G_t(D(\delta_{MA}))$ is given in column seven. The generator from $G_b(d)$ used to obtain $d(1)$ in construction process (4.2) is given in column 8 and the MA-optimal foldover from d used to obtain $d(1)'$ from $d(1)$ is given in column 9. Finally, the number of estimable main effects and two-factor interactions for $G_t(D(\delta_{MA}))$ is given in column 10 and the number of estimable main effects and two-factor interactions for $\tilde{D}$ from (4.2) is given in column 11.

6. Major Findings

For the 16 run blocked FF designs given in Table 1, 11 out of the 38 designs considered yielded semi-foldovers that gave as many estimable main effects and two-factor interactions as the MA-optimal full foldover design. However, the semi-foldover design in 15 other cases in Table 1 yielded a number of estimable main effects and two-factor interactions that were within two of the corresponding MA-optimal full foldover design. We also observe that 15 of the semi-foldover designs $\tilde{D}$ given in Table 1 are saturated, i.e., the number of estimable main effects and two-factor interactions in $\tilde{D}$ is equal to the number 24 of runs in $\tilde{D}$ minus the number of blocks. Finally, we note that any of the designs $\tilde{D}$ given in this table can also be viewed as a possible alternative blocked 2-level FF design to be used in an experimental situation requiring 24 runs.

For the 32 run blocked FF designs given in Table 2, the semi-foldover process is much more effective as 44 out of the 58 designs considered yielded semi-foldovers that gave as many estimable main effects and two-interactions as the MA-optimal full foldover design. In addition, of the remaining 14 designs considered, only three full-foldover designs yielded three or more estimable main effects and two-factor interactions than the corresponding semi-foldover design. We again note that all of the designs $\tilde{D}$ given in this table can be viewed as a possible blocked two-level FF design to be used in an experimental setting where 48 runs are required.

In Table 3, for all 50 64-run designs considered, the semi-foldover combined design yielded as many estimable main effects and two-factor interactions as the full-foldover combined design. This is in line with our general finding that when the number of main effect factors is not large compared to the number of runs, the semi-foldover technique is as useful as the full-foldover for generating additional estimable two-factor interations.

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6. Appendix

Table 1
Feasibility for 16-run blocked FF designs

Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(\mathrm{D}(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects
6-2.1/B1.1	5 = 123 6 = 124	0 3 0 0	$b_1 = 134$	0 4 0 0 0	5, 6, 56	0 1 0 0	134	56	18	18
6-2.2/B1.1	5 = 12 6 = 134	1 1 1 0	$b_1 = 23$	1 2 1 0 0	56	0 0 1 0	23	56	20	19
6.2.2/B1.2	5 = 12 6 = 134	1 1 1 0	$b_1 = 13$	2 1 0 1 0	56	0 0 1 0	13	56	19	18
6-2.4/B1.1	5 = 12 6 = 13	2 1 0 0	$b_1 = 234$	0 2 2 0 0	56	0 1 0 0	234	56	18	18
6-2.1/B2.1	5 = 123 6 = 124	0 3 0 0	$b_1 = 134$ $b_2 = 234$	3 8 0 0 1	5, 6, 56	0 1 0 0	134	56	16	16
6-2.2/B2.1	5 = 12 6 = 134	1 1 1 0	$b_1 = 24$ $b_2 = 1234$	4 5 2 1 0	56	0 0 1 0	1234	56	17	17
6-2.2/B2.2	5 = 12 6 = 134	1 1 1 0	$b_1 = 13$ $b_2 = 14$	6 3 0 3 0	56	0 0 1 0	13	56	15	15
6-2.2/B3.1	5 = 123 6 = 124	0 3 0 0	$b_1 = 13$ $b_2 = 23$ $b_3 = 14$	14 0 12 0 1	5, 6, 56	0 0 1 0	13	56	6	6

(*Continued on next page*)

7-3.1/B1.1	5 = 123 6 = 124 7 = 134	0 7 0 0	$b_1 = 234$	0 7 0 0 1	5, 56, 567 6, 7 56, 67	0 3 0 0 0	234	567	20	20
7-3.1/B1.2	5 = 123 6 = 124 7 = 134	0 7 0 0	$b_1 = 12$	3 0 4 0 1	5, 56, 567 6, 7 56, 67	0 3 0 0 0	12	567	18	16
7-3.2/B1.1	5 = 12 6 = 13 7 = 234	2 3 2 0	$b_1 = 14$	1 4 2 0 1	567	0 1 2 0 0	14	567	24	20
7-3.2/B1.2	5 = 12 6 = 13 7 = 234	2 3 2 0	$b_1 = 24$	2 2 2 2 0	567	0 1 2 0 0	24	567	23	19
7-3.3/B1.1	5 = 12 6 = 13 7 = 24	3 2 1 1	$b_1 = 34$	1 3 3 1 0	567	0 2 0 1 0	34	567	21	20
7-3.3/B1.2	5 = 12 6 = 13 7 = 24	3 2 1 1	$b_1 = 23$	2 3 1 1 1	567	0 2 0 1 0	23	567	21	19
7-3.4/B1.1	5 = 12 6 = 13 7 = 14	3 3 0 0	$b_1 = 234$	0 4 4 0 0	567	0 3 0 0	234	567	20	20
7-3.5/B1.1	5 = 12 6 = 13 7 = 23	4 3 0 0	$b_1 = 1234$	0 3 4 0 0	567	0 3 0 0	1234	567	20	20
7-3.1/B2.1	5 = 123 6 = 124 7 = 134	0 7 0 0	$b_1 = 12$ $b_2 = 13$	9 0 12 0 3 0	5, 56, 567, 6, 7, 57, 67	0 3 0 0	12	567	14	13
7-3.2/B2.1	5 = 12 6 = 13 7 = 234	2 3 2 0	$b_1 = 23$ $b_2 = 1234$	5 10 4 2	567	0 1 2 0	1234	567	21	17

(*Continued on next page*)

Table 1
Feasibility for 16-run blocked FF designs

Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(\mathrm{D}(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects
7-3.2/B2.2	5 = 12 6 = 13 7 = 234	2 3 2 0	$b_1 = 24$ $b_2 = 134$	6 7 6 4	567	0 1 2 0	123	567	20	18
7-3.5/B2.1	5 = 12 6 = 13 7 = 23	4 3 0 0	$b_1 = 14$ $b_2 = 234$	5 8 4 4 3	567	0 3 0 0	234	567	17	17
7-3.1/B3.1	5 = 123 6 = 124 7 = 134	0 7 0 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$	21 0 28 0 7	5, 56, 567, 6, 7, 57, 67	0 3 0 0	12	567	7	7
8-4.1/B1.1	5 = 123 6 = 124 7 = 134 8 = 234	0 14 0 0	$b_1 = 12$	4 0 8 0	5678, 56, 57, 58, 67, 68, 78	0 6 0 0	12	5678	19	17
8-4.2/B1.1	5 = 12 6 = 13 7 = 14 8 = 234	3 7 4 0	$b_1 = 1234$	1 7 4 0	5678	0 3 4 0	1234	5678	27	21
8-4.2/B1.2	5 = 12 6 = 13 7 = 14 8 = 234	3 7 4 0	$b_1 = 23$	3 3 4 4	5678	0 3 4 0	23	5678	26	21

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8-4.4/B1.1	5 = 12 6 = 13 7 = 23 8 = 1234	4 6 4 0 0	$b_1 = 14$	2 4 4 4 2	567	0 3 4 0	14	567	26	21
8-4.6/B1.1	5 = 12 6 = 13 7 = 23 8 = 123	7 7 0 0 1	$b_1 = 14$	1 3 4 4 3	567	0 7 0 0	14	567	21	21
8-4.1/B2.1	5 = 123 6 = 124 7 = 134 8 = 234	0 14 0 0	$b_1 = 12$ $b_2 = 13$	12 0 24 0 12	5678, 56, 57, 58, 67, 68, 78	0 6 0 0	12	5678	15	14
8-4.2/B2.1	5 = 12 6 = 13 7 = 14 8 = 234	3 7 4 0	$b_1 = 23$ $b_2 = 24$	9 9 12 12 3	5678	0 3 4 0	23	5678	22	18
8-4.3/B2.1	5 = 12 6 = 13 7 = 24 8 = 34	4 5 4 2	$b_1 = 124$ $b_2 = 134$	7 14 10 8 7	5678	0 5 0 2	124	5678	20	18
8-4.4/B2.1	5 = 12 6 = 13 7 = 23 8 = 1234	4 6 4 0	$b_1 = 14$ $b_2 = 234$	8 12 8 12	567	0 3 4 0	234	567	22	18
8-4.5/B2.1	5 = 12 6 = 13 7 = 23 8 = 14	5 5 2 2	$b_1 = 24$ $b_2 = 134$	7 13 10 10 7	5678	0 5 0 2	134	5678	20	18

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Table 1
Feasibility for 16-run blocked FF designs

Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(\mathrm{D}(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects
8-4.1/B3.1	$5 = 123$ $6 = 124$ $7 = 134$ $8 = 234$	0 14 0 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$	28 0 56 0 28	5678, 56, 57, 58, 67, 68, 78	0 6 0 0 0	12	5678	8	8
9-5.1/B1.1	$5 = 123$ $6 = 124$ $7 = 134$ $8 = 234$ $9 = 1234$	4 14 8 0	$b_1 = 23$	4 4 8 8 4	5678	0 6 8 0	23	5678	28	18
9-5.2/B1.1	$5 = 12$ $6 = 13$ $7 = 24$ $8 = 34$ $9 = 1234$	6 9 9 6	$b_1 = 23$	3 7 6 6 7	56789	0 9 0 6 0	23	56789	24	21
Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(\mathrm{D}(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects

(*Continued on next page*)

9-5.3/B1.1	5 = 12 6 = 13 7 = 23 8 = 14 9 = 234	6 10 8 4	$b_1 = 1234$	2 8 8 4 6	5678	0 1 0 0 4	1234	5678	23	21
9-5.4/B1.1	5 = 12 6 = 13 7 = 23 8 = 14 9 = 24	7 9 6 6 3 0	$b_1 = 134$	2 7 9 6 4	56789	0 9 0 6 0	134	56789	23	21
9-5.1/B2.1	5 = 123 6 = 124 7 = 134 8 = 234 9 = 1234	4 14 8 0 4	$b_1 = 23$ $b_2 = 24$	12 12 24 24 12	5678	0 6 8 0 0	23	5678	24	18
9-5.2/B2.1	5 = 12 6 = 13 7 = 24 8 = 34 9 = 1234	6 9 9 6 0	$b_1 = 23$ $b_2 = 124$	9 21 18 18 21	56789	0 9 0 6 0	124	56789	20	18

Table 2
Feasibility for 32-run blocked FF designs

Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(D(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects
7-2.1/B1.1	$6 = 123$ $7 = 1245$	0 1 2 0 0	$b_1 = 134$	0 2 2 0 0	6, 67	0 0 1 0 0	134	67	28	28
7-2.1/B1.5	$6 = 123$ $7 = 1245$	0 1 2 0 0	$b_1 = 12$	2 1 0 0 0 1	6, 67	0 0 1 0 0 0	12	67	26	26
7.2.3/B1.1	$6 = 123$ $7 = 124$	0 3 0 0 0	$b_1 = 1345$	0 0 4 0 0 0	6, 7, 67	0 1 0 0 0 0	1345	67	25	25
7-2.4/B1.1	$6 = 12$ $7 = 1345$	1 0 1 1 0	$b_1 = 234$	0 2 2 0 0 0	67	0 0 0 1 0	234	67	28	28
7-2.6/B1.1	$6 = 12$ $7 = 134$	1 1 1 0 0	$b_1 = 235$	0 1 2 1 0 0	67	0 0 1 0 0	235	67	28	28
7-2.8/B1.1	$6 = 12$ $7 = 13$	2 1 0 0 0	$b_1 = 2345$	0 0 2 2 0 0	67	0 1 0 0 0	2345	67	25	25
7-2.1/B2.1	$6 = 123$ $7 = 1245$	0 1 2 0 0	$b_1 = 135$ $b_2 = 2345$	1 6 4 0 1 0	6, 67	0 0 1 0 0 0	2345	67	27	27
7-2.1/B2.3	$6 = 123$ $7 = 1245$	0 1 2 0 0	$b_1 = 134$ $b_2 = 234$	2 5 4 0 0 1	6, 67	0 0 1 0 0 0	134	67	26	26
7-2.3/B2.1	$6 = 123$ $7 = 124$	0 3 0 0 0	$b_1 = 125$ $b_2 = 2345$	0 7 4 0 0 1	6, 7, 67	0 1 0 0 0	2345	67	25	25

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7-2.4/B2.1	6 = 12 7 = 1345	1 0 1 1 0	$b_1 = 235$ $b_2 = 245$	1 5 5 1 0 0	67	0 0 0 1 0 0	235	67	27	27
7-2.5/B2.1	6 = 12 7 = 345	1 1 0 0 1	$b_1 = 134$ $b_2 = 235$	0 6 6 0 0 0	67	0 0 0 0 1	134	67	28	27
7-2.1/B3.1	6 = 123 7 = 1245	0 1 2 0 0 0	$b_1 = 234$ $b_2 = 235$ $b_3 = 1345$	5 12 6 2 3 0	6, 67	0 0 1 0 0 0	234	67	23	23
7-2.2/B3.1	6 = 123 7 = 145	0 2 0 1 0	$b_1 = 135$ $b_2 = 235$ $b_3 = 345$	5 12 5 4 2 0	67	0 0 0 1 0	135	67	23	23
7-2.2/B4.1	6 = 123 7 = 145	0 2 0 1 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$ $b_4 = 15$	21 0 33 0 6 0	67	0 0 0 1 0	12	67	7	7
7-2.3/B4.1	6 = 123 7 = 124	0 3 0 0 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$ $b_4 = 15$	21 0 32 0 7 0	6, 7, 67	0 1 0 0 0	12	67	7	7
8-3.1/B1.1	6 = 123 7 = 124 8 = 1345	0 3 4 0 0 0	$b_1 = 125$	0 3 4 0 0 1 0	6, 7, 67, 68, 78, 678	0 1 2 0 0	125	678	33	33
8-3.1/B1.5	6 = 123 7 = 124 8 = 1345	0 3 4 0 0 0	$b_1 = 13$	2 1 2 2 0 1 0	6, 7	0 1 2 0 0	13	6	31	30
8-3.3/B1.1	6 = 123 7 = 124 8 = 125	0 6 0 0 0 1	$b_1 = 1345$	0 0 8 0 0 0 0	78, 678, 67, 68, 78	0 2 0 0 0 1	1345	678	30	26

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Table 2
Feasibility for 32-run blocked FF designs

Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(D(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects
8-3.4/B1.1	$6 = 123$ $7 = 124$ $8 = 134$	0 7 0 0 0 0	$b_1 = 2345$	0 0 7 0 0 0 1	6, 7, 8, 67, 68, 78, 678	0 3 0 0 0 0	2345	67	28	28
8-3.5/B1.1	$6 = 12$ $7 = 134$ $8 = 235$	1 2 3 1 0 0	$b_1 = 145$	0 3 3 1 1 0 0	678	0 0 2 1 0	145	678	36	36
8-3.7/B1.1	$6 = 12$ $7 = 134$ $8 = 135$	1 3 2 0 1 0	$b_1 = 1234$	1 2 1 2 1	67, 68	0 1 1 0 1	1234	67	32	31
8-3.8/B1.1	$6 = 12$ $7 = 34$ $8 = 135$	2 1 2 2 0 0	$b_1 = 245$	0 2 4 2 0 0	678	0 0 2 1 0 0	245	678	36	36
8-3.1/B2.1	$6 = 123$ $7 = 124$ $8 = 1345$	0 3 4 0 0 0	$b_1 = 125$ $b_2 = 2345$	1 10 8 0 3 2 0	6, 7, 67, 68, 78, 678	0 1 2 0 0	2345	678	32	32
8-3.1/B2.8	$6 = 123$ $7 = 124$ $8 = 1345$	0 3 4 0 0 0	$b_1 = 13$ $b_2 = 14$	6 3 6 6 0 3 0	6, 7, 67, 68, 78, 678	0 1 2 0 0	13	678	28	28
8-3.5/B2.1	$6 = 12$ $7 = 134$ $8 = 235$	1 2 3 1 0 0 0	$b_1 = 145$ $b_2 = 345$	2 8 7 3 3 1 0	678	0 0 2 1 0	145	678	34	34

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8-3.7/B2.1	6 = 12 7 = 134 8 = 135	1 3 2 0 1 0	$b_1 = 1235$ $b_2 = 145$	2 7 6 4 4	678	0 1 0 2 0 0	12345	678	31	31
8-3.1/B3.1	6 = 123 7 = 124 8 = 1345	0 3 4 0 0 0	$b_1 = 234$ $b_2 = 15$ $b_3 = 25$	8 16 11 12 8 0 1	6, 7, 67, 68, 78, 678	0 1 2 0 0	234	678	26	26
8-3.1/B3.2	6 = 123 7 = 124 8 = 1345	0 3 4 0 0 0	$b_1 = 25$ $b_2 = 1235$ $b_3 = 2345$	9 12 16 12 3 4 0	6, 7, 67, 68, 78, 678	0 1 2 0 0	1235	678	25	25
8-3.1/B3.4	6 = 123 7 = 124 8 = 1345	0 3 4 0 0 0	$b_1 = 13$ $b_2 = 23$ $b_3 = 14$	15 6 12 16 16 0	6, 7, 67, 68, 78, 678	0 1 2 0 0	13	678	21	21
8-3.5/B3.1	6 = 12 7 = 134 8 = 235	1 2 3 1 0 0	$b_1 = 15$ $b_2 = 245$ $b_3 = 12345$	7 17 13 9 9 2 0	678	0 0 2 1 0	12345	678	29	29
8-3.8/B3.2	6 = 12 7 = 34 8 = 135	2 1 2 2 0 0	$b_1 = 24$ $b_2 = 1245$ $b_3 = 2345$	8 14 13 14 6 0 1	678	0 0 2 1 0 0	1245	678	28	27
8-3.12/B3.1	6 = 12 7 = 13 8 = 45	3 1 0 2 1 0	$b_1 = 1234$ $b_2 = 25$ $b_3 = 35$	7 15 14 12 7 1 0	678	0 1 0 2 0 0	1235	678	27	27
8-3.2/B4.1	6 = 123 7 = 124 8 = 135	0 5 0 2 0 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$ $b_4 = 15$	28 0 65 0 26 0 1	78	0 1 0 2 0 0	12	78	8	8
8-3.3/B4.1	6 = 123 7 = 124 8 = 125	0 6 0 0 0 1	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$ $b_4 = 15$	28 0 64 0 28 0 0	67, 68, 78	0 2 0 0 0 0	12	67	8	8

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Table 2
Feasibility for 32-run blocked FF designs

Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(\mathrm{D}(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects
9-4.1/B1.1	6 = 123 7 = 124 8 = 125 9 = 1345	0 6 8 0 0 1 0	$b_1 = 2345$	0 4 8 0 0 4 0 0	67, 68, 78	0 2 4 0 0 1 0	2345	67	39	35
9-4.2/B1.1	6 = 123 7 = 124 8 = 134 9 = 2345	0 7 7 0 0 0 1	$b_1 = 15$	1 3 4 4 3 1 0 0	67, 68, 78, 6789	0 3 3 0 0 0 1	15	6789	36	36
9-4.2/B1.3	6 = 123 7 = 124 8 = 134 9 = 2345	0 7 7 0 0 0 1	$b_1 = 12$	3 1 4 4 1 3 0 0	67, 68, 78, 6789	0 3 3 0 0 0 1	12	6789	35	33
9-4.5/B1.1	6 = 123 7 = 124 8 = 134 9 = 234	0 14 0 0 0 1 0	$b_1 = 125$	0 4 0 8 0 4 0 0	67, 68, 69, 78, 79,89, 6789	0 6 0 0 0 1 0	125	6789	30	30
9-4.6/B1.1	6 = 12 7 = 134 8 = 135 9 = 245	1 5 6 2 1 0 0	$b_1 = 345$	0 5 5 2 2 1 1 0	6789	0 1 4 2 0 0 0	345	6789	42	42

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9-4.7/B1.1	6 = 12 7 = 134 8 = 135 9 = 145	1 7 4 0 3 0 0	$b_1 = 2345$	0 3 7 4 0 1 1 0	67, 68, 69, 678, 679, 689	0 3 2 0 2 0 0	2345	678	37	36
9-4.13/B1.1	6 = 12 7 = 13 8 = 14 9 = 2345	3 3 4 4 1 0 0	$b_1 = 1234$	0 5 7 0 0 3 1 0	6789	0 3 1 3 0 0 0	1234	6789	37	37
9-4.20/B1.1	6 = 12 7 = 13 8 = 23 9 = 12345	4 3 3 4 0 0 1	$b_1 = 14$	1 3 4 4 3 1 0 0	678	0 3 0 4 0 0 0	14	678	36	36
9-4.20/B1.2	6 = 12 7 = 13 8 = 23 9 = 12345	4 3 3 4 0 0 1	$b_1 = 123$	3 5 0 0 5 3 0 0	678	0 3 0 4 0 0 0	123	678	36	33
9-4.27/B1.1	6 = 12 7 = 13 8 = 23 9 = 45	5 3 0 4 3 0 0	$b_1 = 1234$	0 3 7 4 0 1 1 0	6789	0 3 0 4 0 0 0	1234	6789	37	37
9-4.1/B2.1	6 = 123 7 = 124 8 = 125 9 = 1345	0 6 8 0 0 1 0	$b_1 = 13$ $b_2 = 2345$	4 8 16 8 4 8 0 0	67, 68, 78	0 2 4 0 0 1 0	2345	67	35	34
9-4.2/B2.1	6 = 123 7 = 124 8 = 134 9 = 2345	0 7 7 0 0 0 1	$b_1 = 15$ $b_2 = 12345$	3 13 8 8 13 3 0 0	67, 68, 78, 6789	0 3 3 0 0 0 1	12345	6789	34	34

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Table 2
Feasibility for 32-run blocked FF designs

Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(\mathrm{D}(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects
9-4.2/B2.2	6 = 123 7 = 124 8 = 134 9 = 2345	0 7 7 0 0 0 1	b_1 = 15 b_2 = 25	5 7 12 12 7 5 0 0	67, 68, 78, 6789	0 3 3 0 0 0 1	15	6789	33	33
9-4.2/B2.3	6 = 123 7 = 124 8 = 134 9 = 2345	0 7 7 0 0 0 1	b_1 = 12 b_2 = 13	9 3 12 12 3 9 0 0	67, 68, 78	0 3 3 0 0 0 1	12	67	31	30
9-4.20/B2.1	6 = 12 7 = 13 8 = 23 9 = 12345	4 3 3 4 0 0 1	b_1 = 14 b_2 = 15	3 9 12 12 9 3 0 0	678	0 3 0 4 0 0 0	123	678	34	34
9-4.20/B2.2	6 = 12 7 = 13 8 = 23 9 = 12345	4 3 3 4 0 0 1	b_1 = 14 b_2 = 234	5 11 8 8 11 5 0 0	678	0 3 0 4 0 0 0	234	678	34	34
9-4.1/B3.1	6 = 123 7 = 124 8 = 125 9 = 1345	0 6 8 0 0 1 0	b_1 = 13 b_2 = 14 b_3 = 2345	12 16 32 24 12 16 0 0	67, 68, 78, 6789	0 2 4 0 0 1 0	2345	6789	29	28

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9-4.2/B3.1	6 = 123 7 = 124 8 = 134 9 = 2345	0 7 7 0 0 1	$b_1 = 15$ $b_2 = 25$ $b_3 = 35$	13 15 28 28 15 13 0 0	67, 68, 78, 6789	0 3 3 0 0 0 1	15	6789	27	27
9-4.2/B3.2	6 = 123 7 = 124 8 = 134 9 = 2345	0 7 7 0 0 1	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$	21 7 28 28 7 21 0 0	67, 68, 78, 6789	0 3 3 0 0 0 1	12	6789	24	24
9-4.3/B3.1	6 = 123 7 = 124 8 = 135 9 = 145	0 9 0 6 0 0 0	$b_1 = 134$ $b_2 = 234$ $b_3 = 12345$	9 27 18 27 21 9 0 1	69, 78, 678, 679, 689, 789	0 3 0 4 0 0 0	12345	678	30	30
9-4.4/B3.1	6 = 123 7 = 124 8 = 134 9 = 125	0 10 0 4 0 1 0	$b_1 = 145$ $b_2 = 245$ $b_3 = 345$	10 24 18 32 18 8 2 0	89	0 3 0 4 0 0 0	145	89	30	30
9-4.20/B3.1	6 = 12 7 = 13 8 = 23 9 = 12345	4 3 3 4 0 0 1	$b_1 = 14$ $b_2 = 234$ $b_3 = 25$	9 23 24 24 23 9 0 0	678	0 3 0 4 0 0 0	234	678	30	30
9-4.3/B4.1	6 = 123 7 = 124 8 = 135 9 = 145	0 9 0 6 0 0 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$ $b_4 = 15$	36 0 117 0 78 0 9 0	69, 78, 678, 679, 689, 789	0 3 0 4 0 0 0	12	678	9	9
9-4.4/B4.1	6 = 123 7 = 124 8 = 134 9 = 125	0 10 0 4 0 1 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$ $b_4 = 15$	36 0 116 0 80 0 80	89	0 3 0 4 0 0 0	12	89	9	9

Table 3
Feasibility for 64-run blocked FF designs

Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(\mathrm{D}(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects
7-1.1/B1.1	7 = 123456	0 0 0 0 1	$b_1 = 123$	0 1 1 0 0 0	7	0 0 0 0 0	123	7	28	28
7-1.2/B1.1	7 = 12345	0 0 0 1 0	$b_1 = 1236$	0 0 2 0 0 0	7	0 0 0 0 0	1236	7	28	28
7-1.3/B1.1	7 = 1234	0 0 1 0 0	$b_1 = 1256$	0 0 1 1 0 0	7	0 0 0 0 0	1256	7	28	28
7-1.4/B1/1	7 = 123	0 1 0 0 0	$b_1 = 12456$	0 0 0 2 0 0	7	0 0 0 0 0	12456	7	28	28
7-1.1/B2.1	7 = 123456	0 0 0 0 1	$b_1 = 123$ $b_2 = 145$	0 3 3 0 0 0	7	0 0 0 0 0	123	7	28	28
7-1.2/B2.1	7 = 12345	0 0 0 1 0	$b_1 = 146$ $b_2 = 2346$	0 3 2 1 0 0	7	0 0 0 0 0	146	7	28	28
7-1.4/B2.1	7 = 123	0 1 0 0 0	$b_1 = 1245$ $b_2 = 1346$	0 0 6 0 0 0	7	0 0 0 0 0	1245	7	28	28
7-1.1/B3.1	7 = 123456	0 0 0 0 1	$b_1 = 123$ $b_2 = 145$ $b_3 = 246$	0 7 7 0 0 0	7	0 0 0 0 0	123	7	28	28
7-1.4/B3.1	7 = 123	0 1 0 0 0	$b_1 = 2345$ $b_2 = 1346$ $b_3 = 456$	0 7 6 0 0 1	7	0 0 0 0 0	2345	7	28	28
7-1.1/B4.1	7 = 123456	0 0 0 0 1	$b_1 = 12$ $b_2 = 13$ $b_3 = 45$ $b_4 = 46$	6 9 9 6 0 0	7	0 0 0 0 0	146	7	22	22

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7-1.2/B4.1	7 = 12345	0 0 0 1 0	$b_1 = 12$ $b_2 = 34$ $b_3 = 135$ $b_4 = 16$	5 12 7 4 2 0	7	0 0 0 0 0	135	7	23	23
7-1.3/B4.1	7 = 1234	0 0 1 0 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 45$ $b_4 = 146$	5 12 7 3 3 0	7	0 0 0 0 0	12	7	23	23
7-1.2/B5.1	7 = 12345	0 0 0 1 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$ $b_4 = 15$ $b_5 = 16$	21 0 34 0 7 0	7	0 0 0 0 0	12	7	7	7
7-1.4/B5.1	7 = 123	0 1 0 0 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$ $b_4 = 15$ $b_5 = 16$	21 0 34 0 7 0	7	0 0 0 0 0	14	7	7	7
8-2.1/B1.1	7 = 1234 8 = 1256	0 0 2 1 0 0	$b_1 = 135$	0 1 2 1 0 0 0	78	0 0 0 1 0 0	135	78	36	36

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Table 3
Feasibility for 64-run blocked FF designs

Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(D(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects
8-2.2/B1.1	$7 = 123$ $8 = 12456$	0 1 0 2 0 0	$b_1 = 1345$	0 0 4 0 0 0 0	7, 78	0 0 0 1 0 0	1345	7	36	36
8-2.4/B1.1	$7 = 123$ $8 = 1245$	0 1 2 0 0 0	$b_1 = 1346$	0 0 2 2 0 0 0	7, 78	0 0 1 0 0 0	1346	7	36	36
8-2.7/B1.1	$7 = 123$ $8 = 124$	0 3 0 0 0 0	$b_1 = 13456$	0 0 0 4 0 0 0	7, 8, 78	0 1 0 0 0 0	13456	7	33	33
8-2.1/B2.1	$7 = 1234$ $8 = 1256$	0 0 2 1 0 0	$b_1 = 135$ $b_2 = 246$	0 4 5 2 1 0 0	78	0 0 0 1 0 0	135	78	36	36
8-2.2/B2.1	$7 = 123$ $8 = 12456$	0 1 0 2 0 0	$b_1 = 145$ $b_2 = 1356$	0 4 4 4 0 0 0	78	0 0 0 1 0 0	145	78	36	36
8-2.3/B2.1	$7 = 123$ $8 = 1456$	0 1 1 0 1 0	$b_1 = 245$ $b_2 = 346$	0 3 6 3 0 0 0	7, 8, 78	0 0 1 0 0 0	245	7	36	36
8-2.5/B2.1	$7 = 123$ $8 = 456$	0 2 0 0 0 1	$b_1 = 1245$ $b_2 = 1346$	0 0 12 0 0 0 0	78	0 0 0 0 0 1	1245	78	36	36
8-2.1/B3.1	$7 = 1234$ $8 = 1256$	0 0 2 1 0 0	$b_1 = 146$ $b_2 = 246$ $b_3 = 13456$	2 8 10 6 1 0 1	78	0 0 0 1 0 0	146	78	34	34
8-2.1/B3.2	$7 = 1234$ $8 = 1256$	0 0 2 1 0 0	$b_1 = 235$ $b_2 = 146$ $b_3 = 2456$	2 9 9 4 3 1 0	78	0 0 0 1 0 0	235	78	34	34

8-2.2/B3.1	7 = 123 8 = 12456	0 1 0 2 0 0	$b_1 = 136$ $b_2 = 2346$ $b_3 = 2356$	1 10 10 4 1 2 0	7, 78	0 0 0 1 0 0	2346	7	35	35
8-2.1/B4.1	7 = 1234 8 = 1256	0 0 2 1 0 0	$b_1 = 13$ $b_2 = 14$ $b_3 = 25$ $b_4 = 26$	7 18 15 10 8 2 0	7, 78	0 0 0 1 0 0	1235	78	29	29
8-2.2/B4.1	7 = 123 8 = 12456	0 1 0 2 0 0	$b_1 = 13$ $b_2 = 14$ $b_3 = 25$ $b_4 = 126$	7 18 14 12 7 2 0	7, 78	0 0 0 1 0 0	126	7	29	29
8-2.3/B4.1	7 = 123 8 = 1456	0 1 1 0 1 0	$b_1 = 12$ $b_2 = 34$ $b_3 = 135$ $b_4 = 136$	7 18 14 11 9 1 0	7, 78	0 0 1 0 0 0	135	7	29	29
8-2.2/5.1	7 = 123 8 = 12456	0 1 0 2 0 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$ $b_4 = 15$ $b_5 = 16$	28 0 69 0 26 0 1	78	0 0 0 1 0 0	12	78	8	8
8-2.5/B5.1	7 = 123 8 = 456	0 2 0 0 0 1	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$ $b_4 = 15$ $b_5 = 16$	28 0 68 0 28 0 0	78	0 0 0 0 0 1	12	78	8	8
9-3.1/B1.1	7 = 123 8 = 1245 9 = 1346	0 1 4 2 0 0 0	$b_1 = 1256$	0 1 4 2 0 1 0 0	7, 78, 79, 789	0 0 2 1 0 0	1256	7	45	45

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Table 3
Feasibility for 64-run blocked FF designs *(Continued)*

Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(\mathrm{D}(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects
9-3.1/B1.11	7 = 123 8 = 1245 9 = 1346	0 1 4 2 0 0 0	b_1 = 23	2 0 1 4 0 0 1 0	78, 79	0 0 2 1 0 0	23	78	43	43
9-3.2/B1.1	7 = 123 8 = 145 9 = 1246	0 2 3 1 1 0 0	b_1 = 356	0 1 3 3 1 0 0 0	78	0 0 1 1 1 0	356	78	45	45
9-3.3/B1.1	7 = 123 8 = 1245 9 = 1246	0 2 4 0 0 1 0	b_1 = 1256	0 0 4 4 0 0 0 0	78, 79	0 0 2 0 0 1	1256	78	45	45
9-3.12/B1.1	7 = 123 8 = 124 9 = 134	0 7 0 0 0 0 0	b_1 = 23456	0 0 0 7 0 0 0 1	7, 8, 9, 78, 79, 89, 789	0 3 0 0 0 0	23456	7	37	37
9-3.1/B2.1	7 = 123 8 = 1245 9 = 1346	0 1 4 2 0 0 0	b_1 = 156 b_2 = 123456	0 6 8 5 4 0 0 1	789	0 0 2 1 0 0	156	789	45	45
9-3.1/B2.2	7 = 123 8 = 1245 9 = 1346	0 1 4 2 0 0 0	b_1 = 135 b_2 = 1256	0 6 9 4 2 2 1 0	79, 789	0 0 2 1 0 0	135	79	45	45
9-3.2/B2.1	7 = 123 8 = 145 9 = 1246	0 2 3 1 1 0 0	b_1 = 156 b_2 = 3456	0 6 8 5 3 1 1 0	789	0 0 2 1 0 0	156	789	45	45
9-3.3/B2.1	7 = 123 8 = 1245 9 = 1246	0 2 4 0 0 1 0	b_1 = 134 b_2 = 23456	0 4 12 4 0 4 0 0	78, 79	0 0 2 0 0 1	134	78	45	45

9-3.6/B2.1	7 = 123 8 = 124 9 = 1356	0 3 2 0 2 0 0	$b_1 = 1346$ $b_2 = 12456$	0 4 11 6 0 2 1 0	79, 89, 789	0 1 1 0 1 0	1346	79	42	42
9-3.1/B3.1	7 = 123 8 = 1245 9 = 1346	0 1 4 2 0 0 0	$b_1 = 126$ $b_2 = 1356$ $b_3 = 23456$	2 14 17 8 8 6 1 0	78, 789	0 0 2 1 0 0	126	78	42	43
9-3.1/B3.3	7 = 123 8 = 1245 9 = 1346	0 1 4 2 0 0 0	$b_1 = 156$ $b_2 = 256$ $b_3 = 3456$	3 13 14 11 11 3 0 1	789	0 0 2 1 0 0	156	789	42	42
9-3.1/B3.15	7 = 123 8 = 1245 9 = 1346	0 1 4 2 0 0 0	$b_1 = 156$ $b_2 = 256$ $b_3 = 356$	6 10 9 16 12 2 1 0	789	0 0 2 1 0 0	156	789	39	39
9-3.2/B3.1	7 = 123 8 = 145 9 = 1246	0 2 3 1 1 0 0	$b_1 = 136$ $b_2 = 2346$ $b_3 = 156$	2 14 16 9 9 5 1 0	789	0 0 2 1 0 0	136	789	43	43
9-3.3/B3.1	7 = 123 8 = 1245 9 = 1246	0 2 4 0 0 1 0	$b_1 = 156$ $b_2 = 256$ $b_3 = 3456$	2 14 16 8 10 6 0 0	78, 79	0 0 2 0 0 1	156	78	43	43
9-3.1/B4.1	7 = 123 8 = 1245 9 = 1346	0 1 4 2 0 0 0	$b_1 = 12$ $b_2 = 134$ $b_3 = 15$ $b_4 = 136$	9 27 26 23 25 9 0 1	78, 789	0 0 2 1 0 0	134	78	36	36
9-3.1/B4.4	7 = 123 8 = 1245 9 = 1346	0 1 4 2 0 0 0	$b_1 = 12$ $b_2 = 13$ $b_3 = 14$ $b_4 = 56$	12 20 25 36 18 4 5 0	78, 789	0 0 2 1 0 0	12	78	33	33

Table 3
Feasibility for 64-run blocked FF designs *(Continued)*

Design	Column(t)	W_t(d)	Column(b)	W_b(d)	δ^*_{MA}	$W_t(\mathrm{D}(\delta^*_{MA}))$	Generator from $G_b(d)$ for $d(1)$	δ^*_{MA} foldover for $d(1)'$	Full foldover estimable effects	Semi-foldover estimable effects
9-3.4/B5.1	$7 = 123$	0 3 0 4 0 0 0	$b_1 = 12$	36 0 123 0	78, 789	0 1 0 2 0 0	12	78	9	9
	$8 = 124$		$b_2 = 13$	80 0 9 0						
	$9 = 13456$		$b_3 = 14$							
			$b_4 = 15$							
			$b_5 = 16$							
9-3.5/B5.1	$7 = 123$	0 3 0 4 0 0 0	$b_1 = 12$	36 0 123 0	789	0 0 0 3 0 0	12	789	9	9
	$8 = 145$		$b_2 = 13$	80 0 9 0						
	$9 = 246$		$b_3 = 14$							
			$b_4 = 15$							
			$b_5 = 16$							
9-3.9/B5.1	$7 = 123$	0 4 0 2 0 1 0	$b_1 = 12$	36 0 122 0	789	0 1 0 2 0 0	12	789	9	9
	$8 = 124$		$b_2 = 13$	82 0 8 0						
	$9 = 156$		$b_3 = 14$							
			$b_4 = 15$							
			$b_5 = 16$							