A Study Guide for
Full, Blocking, and Fractional Factorial Experimental Designs

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Introduction

We recommend our readers use this study guide in accordance with two tremendously wonderful textbooks in experimental design: (Wu and Hamada, 2011) and (Tamhane, 2009).

1 Two-level Full Factorial Design

1.1 Glossary

- *Treatment (a.k.a., run)* refers to a combination of factor levels. E.g. $(A, B, C, D) = (+, +, -, -)$ is a run with factors A and B being $+$ while C and D being $-$;

- *Repitition*: repeat a single run multiple times. E.g. conduct experiment J times for run $(A, B, C, D) = (+, +, -, -)$;

- $Y_{ij} = y_{ij}$, for $i \in \{1, \ldots, 2^k\}$ and $j \in \{1, \ldots, J\}$ is the observed value under j^{th} repetition for i^{th} run;

- 2^k *factorial design*: k factors each at two levels. If all 2^k runs are studied, we call it 2^k full factorial design; see also 2^{k-p} fractional factorial design with k factors and 2^{k-p} runs;

- *Reproducibility* means having a wider inductive basis for conclusions made from factorial experiments;

- *Blocking scheme* b: a scheme which defines the blocks. E.g. $b = (B_1 = 135, B_2 = 235, B_3 = 1234)$ is a blocking scheme that devides a 2^5 design into 2^3 blocks;

- *Confunding*: in the above example, where $b = (B_1 = 135, B_2 = 235, B_3 = 1234)$, we say B_1 is confunded (confused) with the three-way interaction effect, because their estimates are identical;

- *Confunding* (II): for $b = (B_1 = 135, B_2 = 235, B_3 = 1234)$, seven block effects are confunded with seven interactions: $(12, 34, 135, 145, 235, 245, 1234)$. For example, $B_1 \times B_2 = 135 \times 235 = 12(3 * 3)(5 * 5) = 12II = 12$;

- i-*factor* interaction: think of it as i-way interaction. 1-factor interaction is the main effect; 2-factor interaction is the interaction between two factors, etc.;

- $g_i(b)$: the number of $i-factor$ interactons that are confunded with block effect. $g_1(b) = 0$ and $\sum_i g_i(b) = 2^q - 1$;

- *Aberration*: Consider two blocking schemes b_1 and b_2. Denote $r = \arg\min_i \{g_i(b_1) \neq g_i(b2)\}$. Then b_1 has *less aberration* (and hence better) than b_2;

- *Estimability of order* e: all factorial effect of order e are estimable (not confounded) in the blocking scheme;

- *Word*: number of arabic numbers appearing in an i-factor interaction (or blocking). E.g. 125 has *three* words;

1.2 Essential Knowledge

1. *Quadratic loss function*. Define $L(y,t) = c(y-t)^2$, where y is the response and t is a (given) target value. The *risk function* (expected loss) is $\mathbb{E}(L(y,t)) = c\underbrace{\text{Var}y}_{(i)} + c\underbrace{\left(\mathbb{E}y - t\right)^2}_{(ii)}$;

2. *Norminal-the-best Problem* (i) Select levels of <u>some</u> factors to minimize Var$y$; and (ii) Select levels of <u>a</u> factor <u>not</u> in (i) to minimize $|\mathbb{E}y - t|$;

3. The key property of the 2^k factorial design are *balance* and *orthogonality*;

4. *Balance*: each factor level appears in the same number of runs. E.g. in a 2^4 design, $+$ and $-$ appear each 8 times for A, B, C, or D;

5. *Orthogonality*: <u>two</u> factors are orthogonal is all their level combinations appear in the same number of runs. For example, let $A = (-,-,-,-,-,-,-,-)$ and $B = (-,-,-,-,+,+,+,+,-,-,-,-,+,+,+,+)$, then $A^\intercal B = 0$. A <u>design</u> is orthogonal if all pairs of its factors are orthogonal;

6. *Factorial effects* contain *main effect* and *interaction effect*;

7. *Main effect* of factor A: $ME(A) := \bar{z}(A+) - \bar{z}(A-) = \frac{\sum_{\{(i,j)\in\mathcal{A}+\}} y_{ij}}{\#\{\mathcal{A}+\}} - \frac{\sum_{\{(i,j)\in\mathcal{A}-\}} y_{ij}}{\#\{\mathcal{A}-\}}$, where $\bar{z}(A+)$ is the averag of observations at $A+$ and $\bar{z}(A-)$ is the averag of observations at $A-$;

8. *Interaction effect* between factors A and B: $INT(A,B) = \frac{1}{2}\{ME(B|A+) - ME(B|A-)\} = \frac{1}{2}\{ME(A|B+) - ME(A|B-)\}$, where $ME(B|A+) = \bar{z}(B+|A+) - \bar{z}(B-|A+)$;

9. *Multiple-way interaction effect*. $INT(A_1, A_2, \ldots, A_k) = \frac{1}{2}\{INT(A_1, A_2, \ldots, A_{k-1}|A_k+) - INT(A_1, A_2, \ldots, A_{k-1}|A_k-)\} := \bar{z}_+ - \bar{z}_-$;

10. *Blocking effect:* $\bar{y}(II) - \bar{y}(I)$, where $\bar{y}(I)$ and $\bar{y}(II)$ are the average of observations in blocks I and II;

11. *Variance estimation*. Assume the observations for each run are independent and normally distributed with variance σ^2. The factorial effect of $\bar{z}_+ - \bar{z}_-$ as $\hat{\theta}$. $\bar{z}_+$ is the average of $N/2$ observations when $A_k = +$; and $\bar{z}_+$ is the average of $N/2$ observations when $A_k = -$, where $N = 2^k$ for a full unreplicated design, for $N = m2^k$ for a full replicated design with m replicates per run. Then Var$\hat{\theta} = \frac{\sigma^2}{N/2} + \frac{\sigma^2}{N/2} = 4\frac{\sigma^2}{N}$.

1.3 Tests

2 Two-level Fractional Factorial Experiments

2.1 Glossary

- *Motivation*: for a 2^k full factorial design, amongst $2^k - 1$ degrees of freedom, $\sum_{i=3}^{k} \binom{k}{i}$ are used for estimating three-factor and higher order interactions - quite not economic;

- 2^{k-p} *design*: a design with k factors, each at two levels, consisting 2^{k-p} runs (i.e. 2^{-p} fraction);

- But how do we choose 2^{k-p} runs from a 2^k full run design? It turns out p factors have to be "sacrificed". By "sacrificing" p factors, we mean that we shall assign runs to the p factors based on the "independent" $k - p$ factors (by independent, we mean runs assigned to these $k - p$ factors are based on a full factorial design).

  Consider a 2^{6-2} design, where we have factors $1, 2, 3, 4$ independent, and let factors 5 and 6 be, for example, generated by $5 = 12$, and $6 = 134$. Now we the following can be defined:

- *Aliasing*: 5 is *aliased* with 12 interaction, and we denote the aliasing relation as $5 = 12$

or $I = 125$. The notation is the same as defining confounding of a block effect. Aliasing is the price we shall pay for choosing a fraction of the full design;

- *Defining relation*: $I = 125$ and $I = 1346$ are two defining relations of the 2^{6-2} design, where 125 and 1346 are called *defining words*. A 2^{k-p} design has p defining words;

- *Resolution*: the length of the word in a *defining relation* is called *resolution*. For example, the word 125 is of length 3, i.e., of *resolution III*, and the word 1346 is of length 4, i.e., of *resolution IV*;

- *Defining contrast group*: the group formed by the p defining word, namely,

$$I = 125 = 1346 = 23456, \tag{1}$$

  and all aliasing effect of this design can be found using (1) as follows: $I = 125 = 1346 = 23456$, $1 = 25 = 346 = 123456$, $2 = 15 = 12346 = 3456$, ..., $12 = 5 = 2346 = 13456$, ...;

- *Wordlength pattern* of the design: let A_i be the number of words of length i in its defining contrast subgroup. The vector $W = (A_3, \ldots, A_k)$ is called the wordlength pattern;

- *Resolution of the a 2^{k-p} design, R*: the resolution of the a 2^{k-p} design is defined as $R = \min_{r \geq 3}\{r : A_r \geq 1\}$, that is the smallest word length in a *defining contrast group*. In (1), $W = (A_3 = 1, A_4 = 1, A_5 = 1)$, and hence $R = 3$. For a 2^{k-p} design, the larger the *resolution*, the better (see *maximum resolution criterion*, Box and Hunter (1961)), because a lower-resolution design implies aliasing of lower order effects;

- 2_R^{k-p} *design*: a 2_R^{k-p} fractional factorial design with resolution R.

References

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Tamhane, A. C. (2009). *Statistical analysis of designed experiments: theory and applications*, Volume 609. John Wiley & Sons.

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